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NOTE TO A PROBLEM OF T. GALLAI AND G. A. DIRAC

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A counter-example to a conjecture of T. Gallai and G. A. Dirac is given.

Let G be a planar graph and assume that G is 4-*critical,* i.e., $\chi(G)=4$ and $\chi(G')\leqq 3$ for every proper subgraph G' of G (χ denotes the chromatic number). Let $v(x)$ denote the number of edges incident with vertex x. If $v(x)=3$ or $v(x)>3$ then x is called a *secondary* or *primary vertex* of G, respectively.

In his paper [1] T. Gallai mentioned the following:

G. A. Dirac conjectured (oral communication) that every planar 4-*critical graph contains secondary vertices. We believe that each planar* 4-*critical graph must contain at least four secondary vertices.*

This conjecture is contained in the following.

Conjecture. *Each planar* 4-*critical graph with* n *vertices contains at most* $2n-2$ *edges.*

The graph G^* in Fig. 1(2) is a counterexample for these conjectures. It is also a counterexample for a conjecture of H. Grötzsch (communicated by H. Sachs) concerning 3-vertex-colorability [2].

Theorem. G^* *is planar and* 4-*critical,* G^* *contains only primary vertices and* $2n(=80)$ *edges.*

Proof. Since planarity, edge number, absence of secondary vertices, and $\chi(G^*)\geqq 3$ follow immediately from the figures, it suffices to show:

1) $\chi(G^*)>3$.

Assume $\chi(G^*)=3$, then at every 3-coloring of G^* three colors must occur in the central pentagon, two of them (say α, β) twice and the third (say γ) only once. Let vertex a have color γ (circles, see Fig. 1), then the following 13 vertices $b, \ldots, n$ necessarily must have color γ, too. But vertices m, n are adjacent (edge 1) and this contradicts the assumption.

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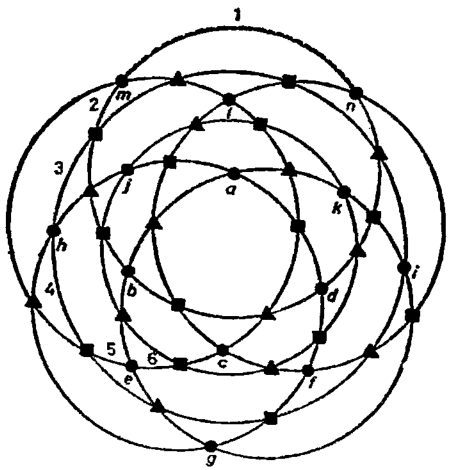

Fig. 1

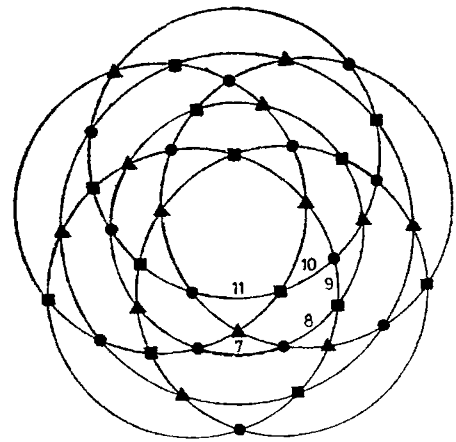

Fig. 2

2) $\chi(G^*-\{e\})=3$ holds for every edge e of G^*.

If we complete the 3-coloring of $G^*-\{1\}$ with color α (triangles) and color β (squares) like in Fig. 1 we see that $\chi(G^*-\{1\})=3$ (and $\chi(G^*)=4$). Moreover, the edge sequence $\{2, 3, 4, 5, 6\}$ is a path with alternating vertex colors β, γ, and if we put color β to vertex m we find that $\chi(G^*-\{2\})=3$. If we continue in changing colors β, γ at this path we find that $\chi(G^*-\{i\})=3$ for $i=1, \dots, 6$. In Fig. 2 we have a 3-coloring of $G^*-\{7\}$. The same argument as above leads to $\chi(G^*-\{j\})=3$ for $j=7, 8, 9, 10, 11$. From the symmetry of G^* follows that $\chi(G^*-\{e\})=3$ holds for every edge e of G^*. ∎

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References

[1] T. GALLAI, Critical graphs, in: *Theory of Graphs and its Applications* (Proc. Symp. Smolenice 1963), Publ. House Czechoslovak *Acad. Sci.*, Prague 1964, 43—45.

[2] G. KOESTER, Bemerkung zu einem Problem von H. Grötzsch. *Wiss. Z. Univ. Halle,* **XXXIII** (1984), M.H.5. S.129.

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